9.2 General Fourier Series and Convergence

$f(x)$ period 2π, given on $-\pi < x < \pi$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

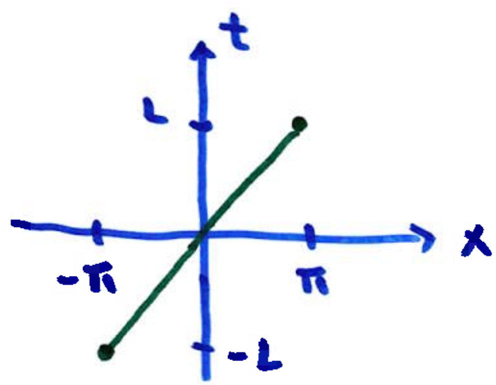
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

let's relax the period $= 2\pi$ part

let the period be $2L$ (L is the half period)

the function is given on $-L < t < L$

define $t = \frac{L}{\pi}x$



when $x = -\pi$ $t = -L$

$x = \pi$ $t = L$

$x = \frac{\pi}{L} t$ so $dx = \frac{\pi}{L} dt$ the formulas we had become

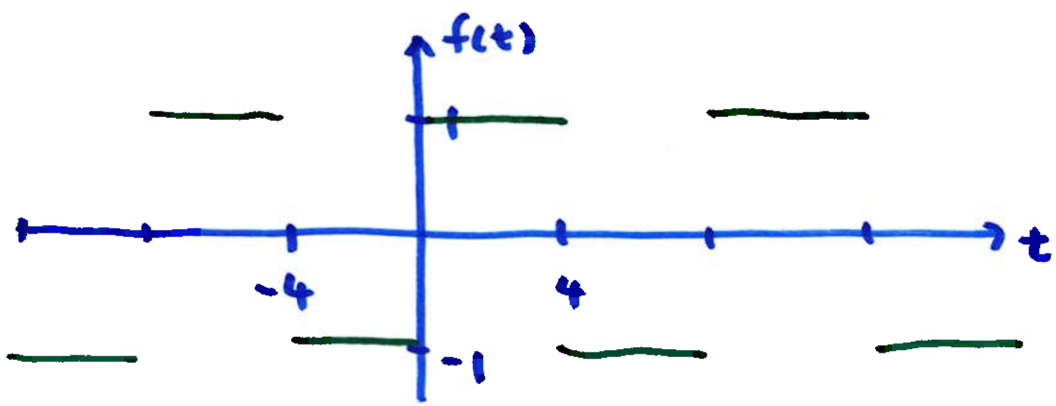
$$\boxed{f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right) \right]}$$

$$a_n = \frac{1}{\cancel{\pi}} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) \cdot \frac{\cancel{\pi}}{L} dt$$

$$\boxed{\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) dt \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi}{L} t\right) dt \end{aligned}}$$

example

$$f(t) = \begin{cases} -1 & -4 < t < 0 \\ 1 & 0 < t < 4 \end{cases} \qquad \text{period } 8 \quad (L = 4)$$



$$a_0 = \frac{1}{4}\underbrace{\int_{-4}^{4} f(t)\,dt}_{\text{net area under } f(t)} = 0$$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(t)\cos\left(\frac{n\pi}{4}t\right)dt$$

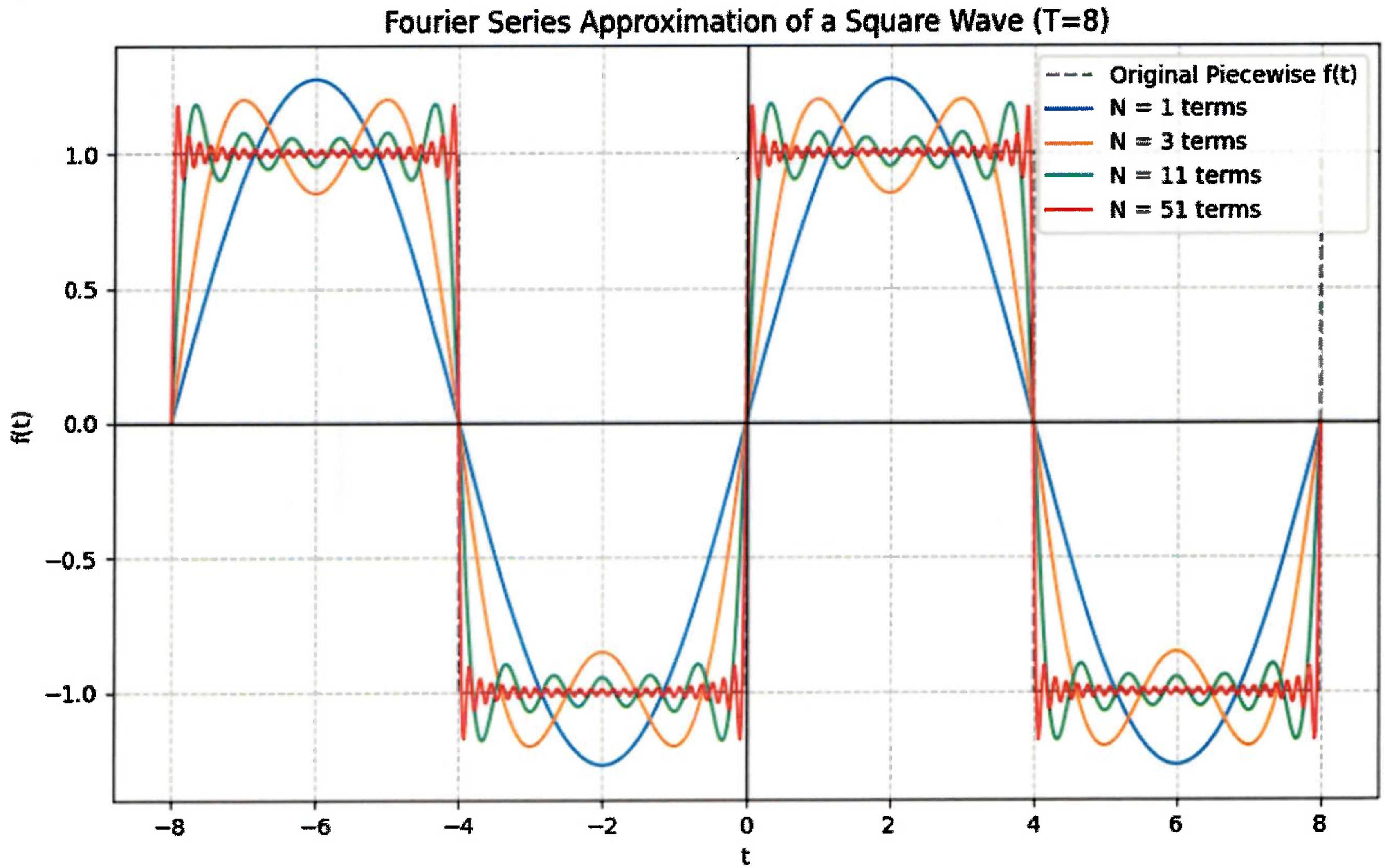
$$= \cdots = 0 \qquad \text{no cosine terms}$$

$$b_n = \frac{1}{4}\int_{-4}^{4} f(t)\sin\left(\frac{n\pi}{4}t\right)dt$$

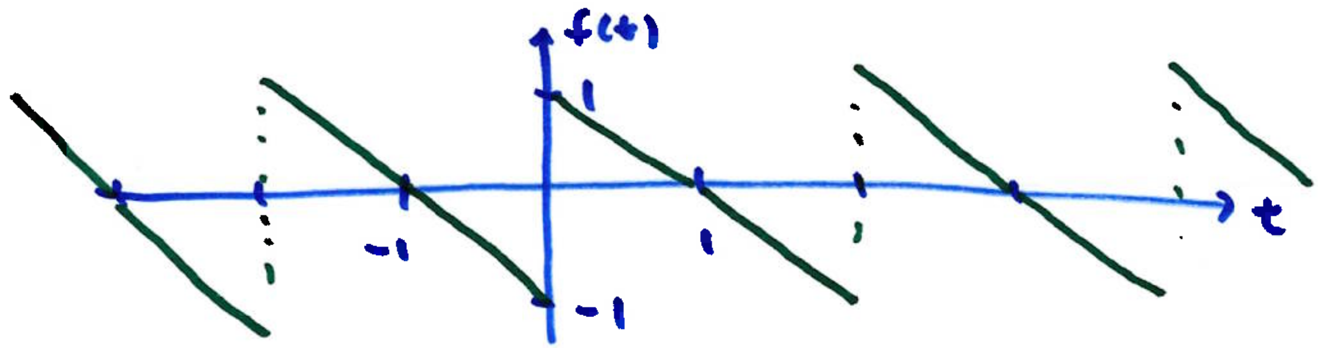
$$= \cdots = \frac{2}{n\pi}\left[1-\cos(n\pi)\right] = \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \frac{4}{n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi}{4}t\right)$$

$$\sim \sum_{n \text{ odd}}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi}{4}t\right) \sim \frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi}{4}t\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi}{4}t\right)$$


Fourier Series Approximation of a Square Wave (T=8)
Original Piecewise f(t)
N = 1 terms
N = 3 terms
N = 11 terms
N = 51 terms
1.0
0.5
0.0
−0.5
−1.0
f(t)
−8
−6
−4
−2
0
2
4
6
8
t

$$f(t) = \begin{cases} -1-t & -1<t<0 \\ 1-t & 0<t<1 \end{cases} \qquad \text{period } 2 \quad (L=1)$$

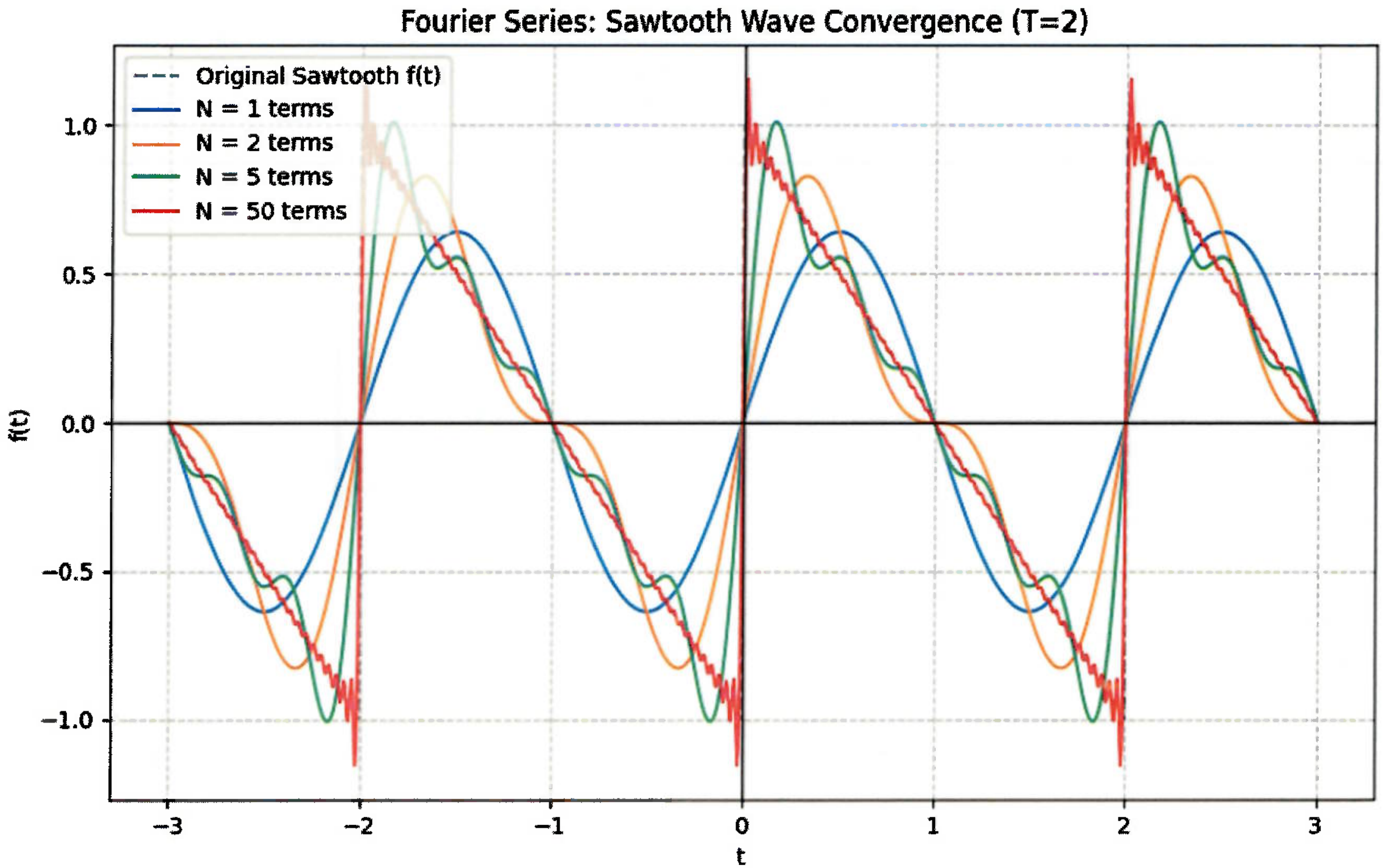


$$a_0 = \frac{1}{1}\int_{-1}^{1} f(t)\,dt = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos(n\pi t)\,dt = \cdots = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin(n\pi t)\,dt = \cdots = \frac{2}{n\pi}$$

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi t) \sim \frac{2}{\pi}\sin(\pi t) + \frac{2}{2\pi}\sin(2\pi t) + \frac{2}{3\pi}\sin(3\pi t) + \cdots$$

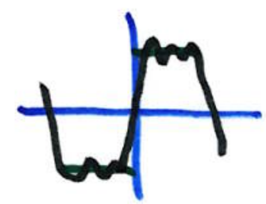

Fourier Series: Sawtooth Wave Convergence (T=2)
Original Sawtooth f(t)
N = 1 terms
N = 2 terms
N = 5 terms
N = 50 terms
1.0
0.5
0.0
−0.5
−1.0
f(t)
−3
−2
−1
0
1
2
3
t

Two things we notice: Fourier series cuts through the middle (average) at discontinuity

$$\frac{f(t^-)+f(t^+)}{2}$$

why?

1st example



$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \underbrace{\sin\left(\frac{n\pi t}{4}\right)}_{0 \text{ at } t=0}$$

so the whole series goes to 0

$$\frac{-1+1}{2}=0$$

2nd : the overshoot before discontinuity

the overshoot does NOT go away as n increases
(gets closer to discontinuity)

it's always $\approx 9\%$ of jump

$\rightarrow$ Gibbs phenomenon

we can hear this artifact in "low-fi" or 8-bit sound